PHYSICS (PRE-STAGE LEVEL)

LECTURE 14

Complex Numbers

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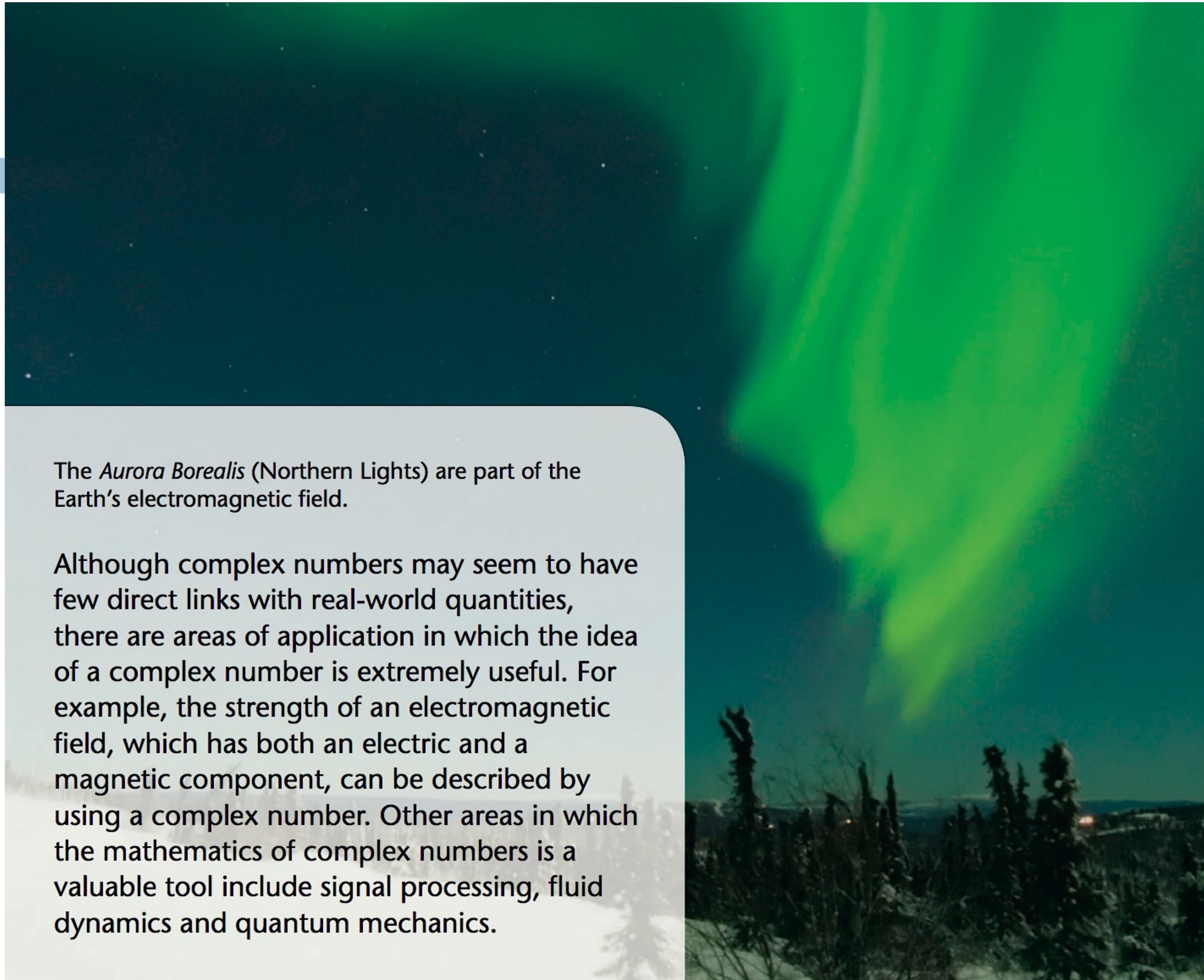


The *Aurora Borealis* (Northern Lights) are part of the Earth's electromagnetic field.

Although complex numbers may seem to have few direct links with real-world quantities, there are areas of application in which the idea of a complex number is extremely useful. For example, the strength of an electromagnetic field, which has both an electric and a magnetic component, can be described by using a complex number. Other areas in which the mathematics of complex numbers is a valuable tool include signal processing, fluid dynamics and quantum mechanics.

Part I

You can use real and imaginary numbers.

For the equation $ax^2 + bx + c = 0$, the discriminant is $b^2 - 4ac$.
If $b^2 - 4ac > 0$, there are two different real roots.
If $b^2 - 4ac = 0$, there are two equal real roots.
If $b^2 - 4ac < 0$, there are no real roots.

In the case $b^2 - 4ac < 0$, the problem is that you reach a situation where you need to find the square root of a negative number, which is not 'real'.

To solve this problem, another type of number called an 'imaginary number' is used.

The 'imaginary number' $\sqrt{(-1)}$ is called i (or sometimes j in electrical engineering), and sums of real and imaginary numbers, such as $3 + 2\text{i}$, are known as **complex numbers**.

- **A complex number is written in the form $a + b\text{i}$.**
- **You can add and subtract complex numbers.**
- $\sqrt{(-1)} = \text{i}$
- **An imaginary number is a number of the form $b\text{i}$, where b is a real number ($b \in \mathbb{R}$).**

Example 1

Write $\sqrt{(-36)}$ in terms of i.

$$\sqrt{(-36)} = \sqrt{(36 \times -1)} = \sqrt{36}\sqrt{(-1)} = 6i$$

Example 2

Write $\sqrt{(-28)}$ in terms of i.

$$\sqrt{(-28)} = \sqrt{(28 \times -1)} = \sqrt{28}\sqrt{(-1)} = \sqrt{4}\sqrt{7}\sqrt{(-1)} = 2\sqrt{7}i \text{ or } 2i\sqrt{7} \text{ or } (2\sqrt{7})i$$

This can be written as $2i\sqrt{7}$ or $(2\sqrt{7})i$ to avoid confusion with $2\sqrt{7i}$.

Example 3

Solve the equation $x^2 + 9 = 0$.

$$x^2 = -9$$

$$x = \pm\sqrt{(-9)} = \pm\sqrt{(9 \times -1)} = \pm\sqrt{9}\sqrt{(-1)} = \pm 3i$$

$$x = \pm 3i \qquad (x = +3i,\ x = -3i)$$

Note that just as $x^2 = 9$ has two roots $+3$ and -3, $x^2 = -9$ also has two roots $+3i$ and $-3i$.

- A complex number is a number of the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
- For the complex number $a + bi$, a is called the **real part** and b is called the **imaginary part**.
- The complete set of complex numbers is called $\mathbb{C}$.

Example 4

Solve the equation $x^2 + 6x + 25 = 0$.

Method 1 (Completing the square)

$$x^2 + 6x = (x + 3)^2 - 9$$

$$x^2 + 6x + 25 = (x + 3)^2 - 9 + 25 = (x + 3)^2 + 16$$

$$(x + 3)^2 + 16 = 0$$

$$(x + 3)^2 = -16$$

$$x + 3 = \pm\sqrt{(-16)} = \pm 4\mathrm{i}$$

$$x = -3 \pm 4\mathrm{i}$$

$$x = -3 + 4\mathrm{i}, \qquad x = -3 - 4\mathrm{i}$$

Because
$(x + 3)^2 = (x + 3)(x + 3)$
$= x^2 + 6x + 9$

$\sqrt{(-16)} = \sqrt{(16 \times -1)}$
$= \sqrt{16}\sqrt{(-1)} = 4\mathrm{i}$

Method 2 (Quadratic formula)

$$x = \frac{-6 \pm \sqrt{(6^2 - 4 \times 1 \times 25)}}{2} = \frac{-6 \pm \sqrt{(-64)}}{2}$$

$$\sqrt{(-64)} = \pm 8i$$

$$x = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$x = -3 + 4i, \qquad x = -3 - 4i$$

Using

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$\sqrt{(-64)} = \sqrt{(64 \times -1)}$$
$$= \sqrt{64}\,\sqrt{(-1)} = 8i$$

- **In a complex number, the real part and the imaginary part cannot be combined to form a single term.**
- **You can add complex numbers by adding the real parts and adding the imaginary parts.**
- **You can subtract complex numbers by subtracting the real parts and subtracting the imaginary parts.**

Example 5

Simplify, giving your answer in the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

7 **a** $(2 + 5i) + (7 + 3i)$ **b** $(3 - 4i) + (-5 + 6i)$ **c** $2(5 - 8i)$

d $(1 + 8i) - (6 + i)$ **e** $(2 - 5i) - (5 - 11i)$ **f** $(2 + 3i) - (2 - 3i)$

a $(2 + 5i) + (7 + 3i) = (2 + 7) + i(5 + 3) = 9 + 8i$

Add real parts and add imaginary parts.

b $(3 - 4i) + (-5 + 6i) = (3 - 5) + i(-4 + 6) = -2 + 2i$

c $2(5 - 8i) = 10 - 16i$

This is the same as $(5 - 8i) + (5 - 8i)$

d $(1 + 8i) - (6 + i) = (1 - 6) + i(8 - 1) = -5 + 7i$

Subtract real parts and subtract imaginary parts.

e $(2 - 5i) - (5 - 11i) = (2 - 5) + i(-5 - (-11)) = -3 + 6i$

f $(2 + 3i) - (2 - 3i) = (2 - 2) + i(3 - (-3)) = 6i$

The answer has no real part. This is called purely imaginary.

You can multiply complex numbers and simplify powers of i.

- **You can multiply complex numbers using the same technique as you use for multiplying brackets in algebra, and you can simplify powers of i.**
- **Since $i = \sqrt{(-1)}$, $i^2 = -1$**

Example 6

Multiply $(2 + 3i)$ by $(4 + 5i)$

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2(4 + 5i) + 3i(4 + 5i)\\ &= 8 + 10i + 12i + 15i^2\\ &= 8 + 10i + 12i - 15\\ &= (8 - 15) + (10i + 12i)\\ &= -7 + 22i\end{aligned}$$

Multiply the two brackets as you would in algebra.

Use the fact that $i^2 = -1$.

Add real parts and add imaginary parts.

Example 7

Express $(7 - 4i)^2$ in the form $a + bi$.

$$
\begin{aligned}
(7 - 4i)(7 - 4i) &= 7(7 - 4i) - 4i(7 - 4i) \\
&= 49 - 28i - 28i + 16i^2 \\
&= 49 - 28i - 28i - 16 \\
&= (49 - 16) + (-28i - 28i) \\
&= 33 - 56i
\end{aligned}
$$

Multiply the two brackets as you would in algebra.

Use the fact that $i^2 = -1$.

Add real parts and add imaginary parts.

Example 8

Simplify $(2-3i)(4-5i)(1+3i)$

$(2-3i)(4-5i) = 2(4-5i) - 3i(4-5i)$

$= 8 - 10i - 12i + 15i^2 = 8 - 10i - 12i - 15 = -7 - 22i$

$(-7-22i)(1+3i) = -7(1+3i) - 22i(1+3i)$

$= -7 - 21i - 22i - 66i^2 = 59 - 43i$

First multiply two of the brackets.

Then multiply the result by the third bracket.

Example 9

Simplify

a i^3 **b** i^4 **c** $(2i)^5$

a $i^3 = i \times i \times i = i^2 \times i = -i$

b $i^4 = i \times i \times i \times i = i^2 \times i^2 = -1 \times -1 = 1$

c $(2i)^5 = 2i \times 2i \times 2i \times 2i \times 2i = 32(i \times i \times i \times i \times i)$

$= 32(i^2 \times i^2 \times i) = 32 \times -1 \times -1 \times i = 32i$

$i^2 = -1$

First multiply the 2s (2^5).

You can find the complex conjugate of a complex number.

- You can write down the complex conjugate of a complex number, and you can divide two complex numbers by using the complex conjugate of the denominator.
- The complex number $a - bi$ is called the complex conjugate of the complex number $a + bi$.
- The complex numbers $a + bi$ and $a - bi$ are called a **complex conjugate pair.**
- The complex conjugate of z is called z^*, so if $z = a + bi$, $z^* = a - bi$.

Example 10

Write down the complex conjugate of

a $2 + 3i$ **b** $5 - 2i$ **c** $\sqrt{3} + i$ **d** $1 - i\sqrt{5}$

a $2 - 3i$ **b** $5 + 2i$

c $\sqrt{3} - i$ **d** $1 + i\sqrt{5}$

Just change the sign of the imaginary part (from + to –, or – to +).

Example 11

Find $z + z^*$ and zz^*, given that

a $z = 3 + 5\text{i}$ **b** $z = 2 - 7\text{i}$ **c** $z = 2\sqrt{2} + \text{i}\sqrt{2}$

a

$$z^* = 3 - 5\text{i}$$

$$z + z^* = (3 + 5\text{i}) + (3 - 5\text{i}) = (3 + 3) + \text{i}(5 - 5) = 6$$

Note that $z + z^*$ is real.

$$zz^* = (3 + 5\text{i})(3 - 5\text{i}) = 3(3 - 5\text{i}) + 5\text{i}(3 - 5\text{i})$$

$$= 9 - 15\text{i} + 15\text{i} - 25\text{i}^2 = 9 + 25 = 34$$

Note that zz^* is real.

b

$$z^* = 2 + 7\text{i}$$

$$z + z^* = (2 - 7\text{i}) + (2 + 7\text{i}) = (2 + 2) + \text{i}(-7 + 7) = 4$$

Note that $z + z^*$ is real.

$$zz^* = (2 - 7\text{i})(2 + 7\text{i}) = 2(2 + 7\text{i}) - 7\text{i}(2 + 7\text{i})$$

$$= 4 + 14\text{i} - 14\text{i} - 49\text{i}^2 = 4 + 49 = 53$$

Note that zz^* is real.

c

$$z^* = 2\sqrt{2} - \text{i}\sqrt{2}$$

$$z + z^* = (2\sqrt{2} + \text{i}\sqrt{2}) + (2\sqrt{2} - \text{i}\sqrt{2})$$

$$= (2\sqrt{2} + 2\sqrt{2}) + \text{i}(\sqrt{2} - \sqrt{2}) = 4\sqrt{2}$$

Note that $z + z^*$ is real.

$$zz^* = (2\sqrt{2} + \text{i}\sqrt{2})(2\sqrt{2} - \text{i}\sqrt{2})$$

$$= 2\sqrt{2}(2\sqrt{2} - \text{i}\sqrt{2}) + \text{i}\sqrt{2}(2\sqrt{2} - \text{i}\sqrt{2})$$

$$= 8 - 4\text{i} + 4\text{i} - 2\text{i}^2 = 8 + 2 = 10$$

Note that zz^* is real.

Example 12

Simplify $(10 + 5i) \div (1 + 2i)$

$$(10 + 5i) \div (1 + 2i) = \frac{10 + 5i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i}$$

The complex conjugate of the denominator is $1 - 2i$. Multiply numerator and denominator by this.

$$\frac{10 + 5i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} = \frac{(10 + 5i)(1 - 2i)}{(1 + 2i)(1 - 2i)}$$

$$\begin{aligned}(10 + 5i)(1 - 2i) &= 10(1 - 2i) + 5i(1 - 2i)\\ &= 10 - 20i + 5i - 10i^2\\ &= 20 - 15i\end{aligned}$$

$$\begin{aligned}(1 + 2i)(1 - 2i) &= 1(1 - 2i) + 2i(1 - 2i)\\ &= 1 - 2i + 2i - 4i^2 = 5\end{aligned}$$

$$(10 + 5i) \div (1 + 2i) = \frac{20 - 15i}{5} = 4 - 3i$$

Divide each term in the numerator by 5.

Example 13

Simplify $(5 + 4i) \div (2 - 3i)$

$$(5 + 4i) \div (2 - 3i) = \frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$

The complex conjugate of the denominator is $2 + 3i$. Multiply numerator and denominator by this.

$$\frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i} = \frac{(5 + 4i)(2 + 3i)}{(2 - 3i)(2 + 3i)}$$

$$(5 + 4i)(2 + 3i) = 5(2 + 3i) + 4i(2 + 3i)$$
$$= 10 + 15i + 8i + 12i^2$$
$$= -2 + 23i$$

$$(2 - 3i)(2 + 3i) = 2(2 + 3i) - 3i(2 + 3i)$$
$$= 4 + 6i - 6i - 9i^2 = 13$$

$$(5 + 4i) \div (2 - 3i) = \frac{-2 + 23i}{13} = -\frac{2}{13} + \frac{23}{13}i$$

Divide each term in the numerator by 13.

The division process shown in Examples 12 and 13 is similar to the process used to divide surds.

For surds the denominator is rationalised. For complex numbers the denominator is made real.

- **If the roots α and β of a quadratic equation are complex, α and β will always be a complex conjugate pair.**

- **If the roots of the equation are α and β, the equation is $(x - \alpha)(x - \beta) = 0$**
  $(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 - (\alpha + \beta)x + \alpha\beta$

Example 14

Find the quadratic equation that has roots $3 + 5i$ and $3 - 5i$.

For this equation $\alpha + \beta = (3 + 5i) + (3 - 5i) = 6$

and $\alpha\beta = (3 + 5i)(3 - 5i) = 9 + 15i - 15i - 25i^2 = 34$

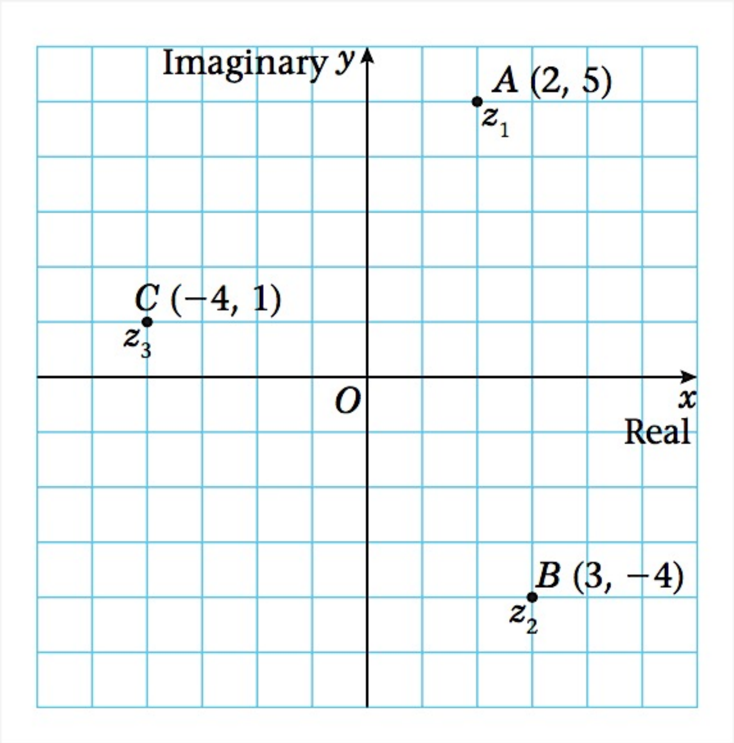
The equation is $x^2 - 6x + 34 = 0$

You can represent complex numbers on an Argand diagram.

- You can represent **complex numbers** on a diagram, called an **Argand diagram**.
- A **real number** can be represented as a point on a **straight line (a number line, which has one dimension)**.
- A **complex number**, having two components (real and imaginary), can be represented as a point in a **plane (two dimensions)**.
- The complex number $z = x + iy$ is represented by the point (x, y), where x and y are **Cartesian coordinates.**
- The Cartesian coordinate diagram used to represent complex numbers is called an Argand diagram.
- The x-axis in the Argand Diagram is called the **real axis** and the y-axis is called the **imaginary axis.**

Example 15

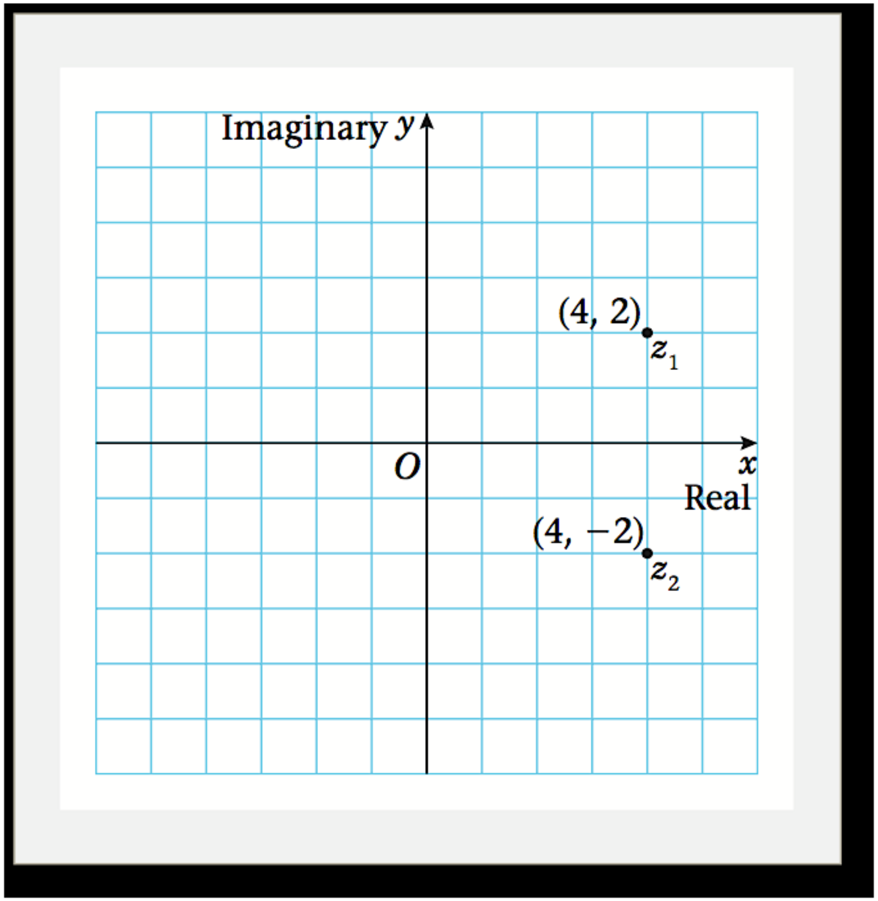
The complex numbers $z_1 = 2 + 5\mathrm{i}$, $z_2 = 3 - 4\mathrm{i}$ and $z_3 = -4 + \mathrm{i}$ are represented by the points A, B and C respectively on an Argand diagram. Sketch the Argand diagram.



For $z_1 = 2 + 5\mathrm{i}$, plot $(2, 5)$.
For $z_2 = 3 - 4\mathrm{i}$, plot $(3, -4)$.
For $z_3 = -4 + \mathrm{i}$, plot $(-4, 1)$.

Example 16

Show the complex conjugates $z_1 = 4 + 2\text{i}$ and $z^* = 4 - 2\text{i}$ on an Argand diagram.

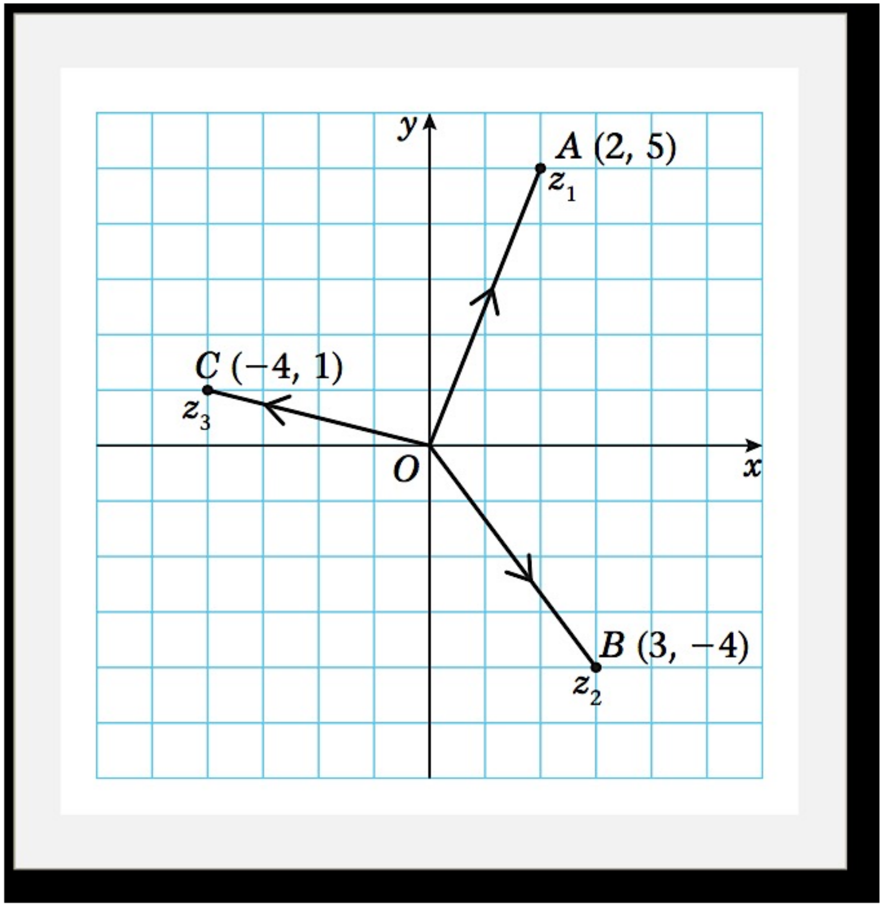


Note that complex conjugates will always be placed symmetrically above and below the real axis.

The complex number $z = x + \text{i}y$ can also be represented by the vector $\overrightarrow{OP}$, where O is the origin and P is the point (x, y) on the Argand diagram.

Example 17

Show the complex numbers $z_1 = 2 + 5\text{i}$, $z_2 = 3 - 4\text{i}$ and $z_3 = -4 + \text{i}$ on an Argand diagram.

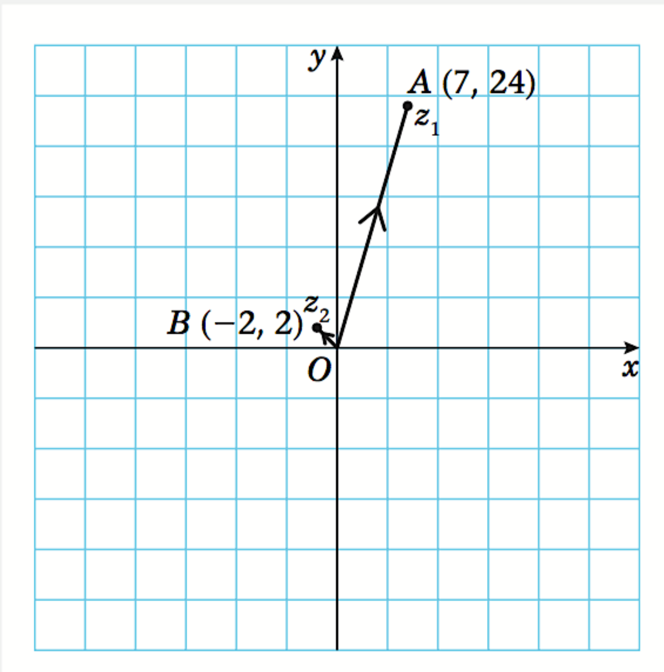


For $z_1 = 2 + 5\text{i}$, show the vector from (0, 0) to (2, 5).
Similarly for z_2 and z_3.

If you label the diagram with letters A, B and C, make sure that you show which letter represents which vector.

Example 18

The complex numbers $z_1 = 7 + 24\text{i}$ and $z_2 = -2 + 2\text{i}$ are represented by the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively on an Argand diagram (where O is the origin). Draw the diagram and calculate $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$.



$$|\overrightarrow{OA}| = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$$

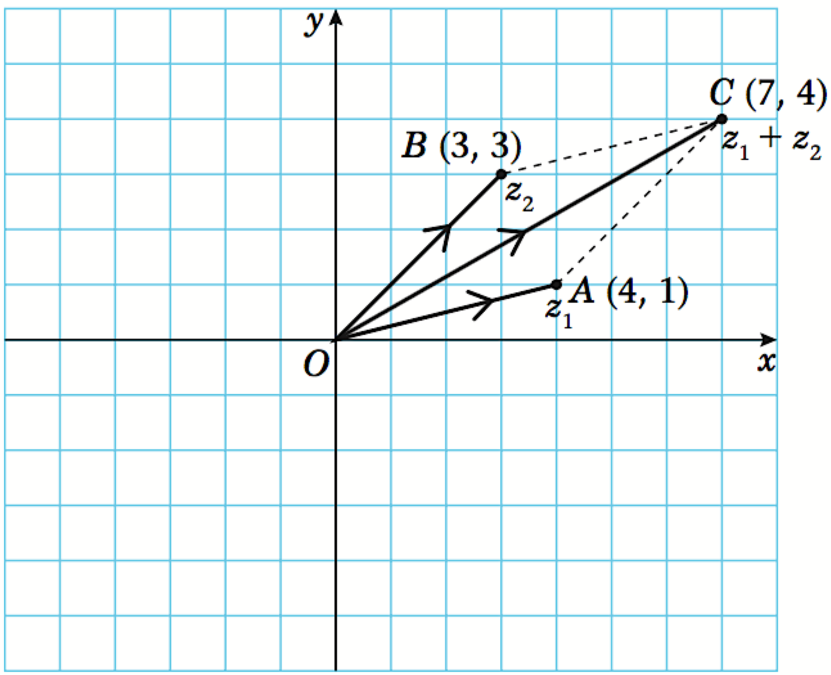
$$|\overrightarrow{OB}| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

■ **Addition of complex numbers can be represented on the Argand diagram by the addition of their respective vectors on the diagram.**

Example 19

21 $z_1 = 4 + i$ and $z_2 = 3 + 3i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

$z_1 + z_2 = (4 + 3) + i(1 + 3) = 7 + 4i$

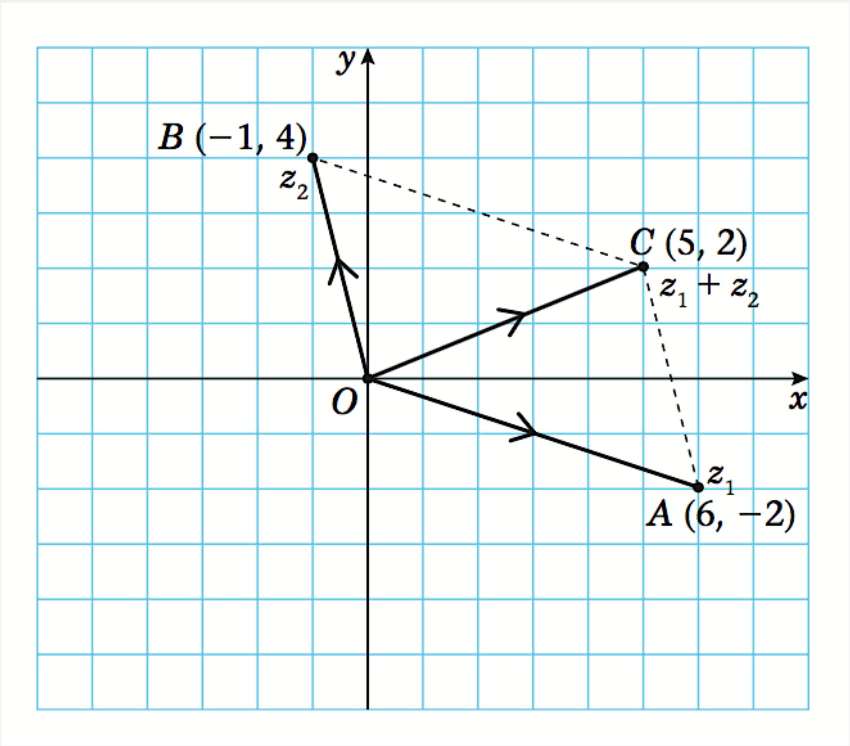


Note that the vector for $z_1 + z_2$ ($\overrightarrow{OC}$) is the diagonal of the parallelogram. This is because $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB}$.

Example 20

$z_1 = 6 - 2i$ and $z_2 = -1 + 4i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

$z_1 + z_2 = (6 - 1) + i(-2 + 4) = 5 + 2i$

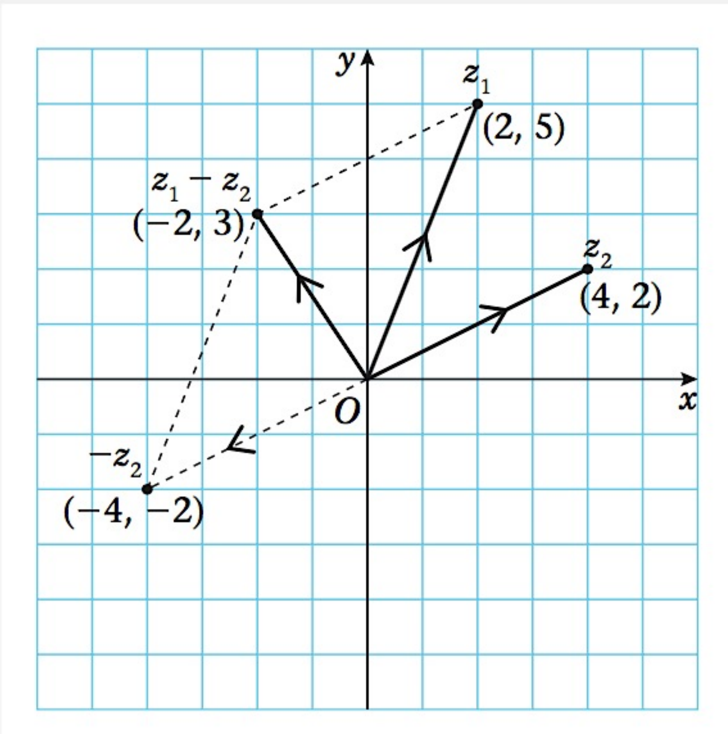


Note that the vector for $z_1 + z_2$ ($\overrightarrow{OC}$) is the diagonal of the parallelogram. This is because $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB}$.

Example 21

$z_1 = 2 + 5i$ and $z_2 = 4 + 2i$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.

$$z_1 - z_2 = (2 - 4) + i(5 - 2) = -2 + 3i$$



$z_1 - z_2 = z_1 + (-z_2)$.
The vector for $-z_2$ is shown by the dotted line on the diagram.

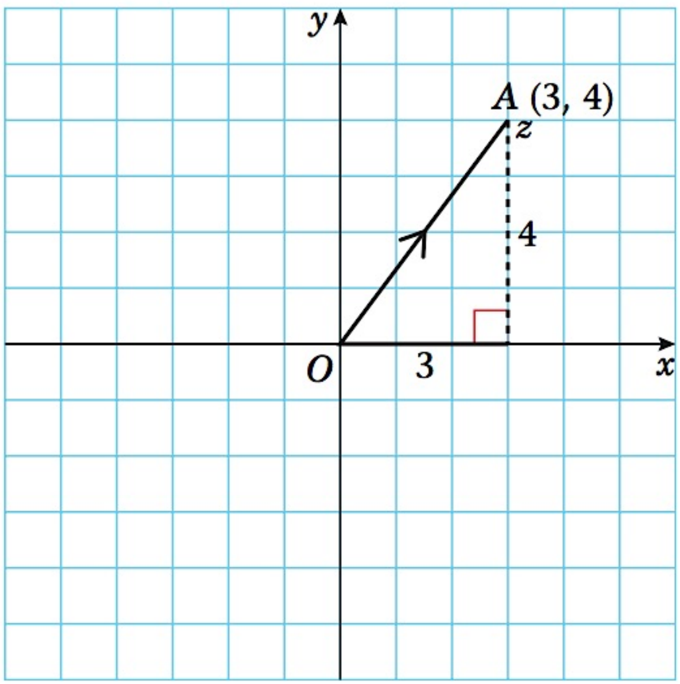
You can find the value of *r*, the modulus of a complex number *z*, and the value of *θ*, the argument of *z*.

- Consider the complex number $3 + 4\text{i}$, represented on an Argand diagram by the point A, or by the vector $\overrightarrow{OA}$.

  The length OA or $|\overrightarrow{OA}|$, the magnitude of vector $|\overrightarrow{OA}|$, is found by Pythagoras' theorem:

  $|\overrightarrow{OA}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

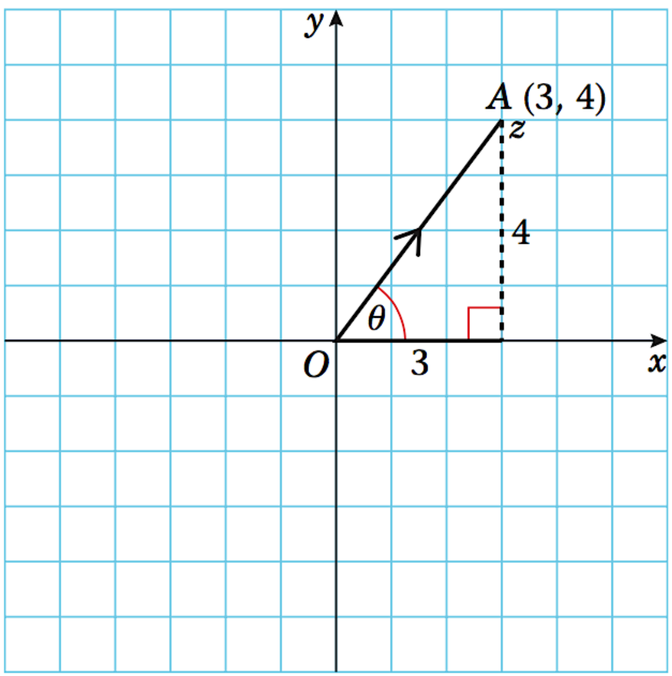
  This number is called the modulus of the complex number $3 + 4\text{i}$.



- The **modulus** of the complex number $z = x + iy$ is given by $\sqrt{x^2 + y^2}$.
- The modulus of the complex number $z = x + iy$ is written as r or $|z|$ or $|x + iy|$, so $r = \sqrt{x^2 + y^2}$.
- $|z| = \sqrt{x^2 + y^2}$.
- $|x + iy| = \sqrt{x^2 + y^2}$.

■ **The modulus of any non-zero complex number is positive.**

Consider again the complex number $z = 3 + 4i$.



By convention, angles are measured from the positive x-axis (or the positive real axis), anticlockwise being positive.

The angle θ shown on the Argand diagram, measured from the positive real axis, is found by trigonometry:

$\tan \theta = \frac{4}{3}$,

$\theta = \arctan \frac{4}{3} \approx 0.927$ radians

This angle is called the argument of the complex number $3 + 4i$.

- The **argument** of the complex number $z = x + \mathrm{i}y$ is the angle θ between the positive real axis and the vector representing z on the Argand diagram.

- For the argument θ of the complex number $z = x + \mathrm{i}y$, $\tan\theta = \frac{y}{x}$.

- The argument θ of any complex number is such that $-\pi < \theta \leqslant \pi$ (or $-180° < \theta \leqslant 180°$). (This is sometimes referred to as the **principal argument**).

- The argument of a complex number z is written as **arg** z.

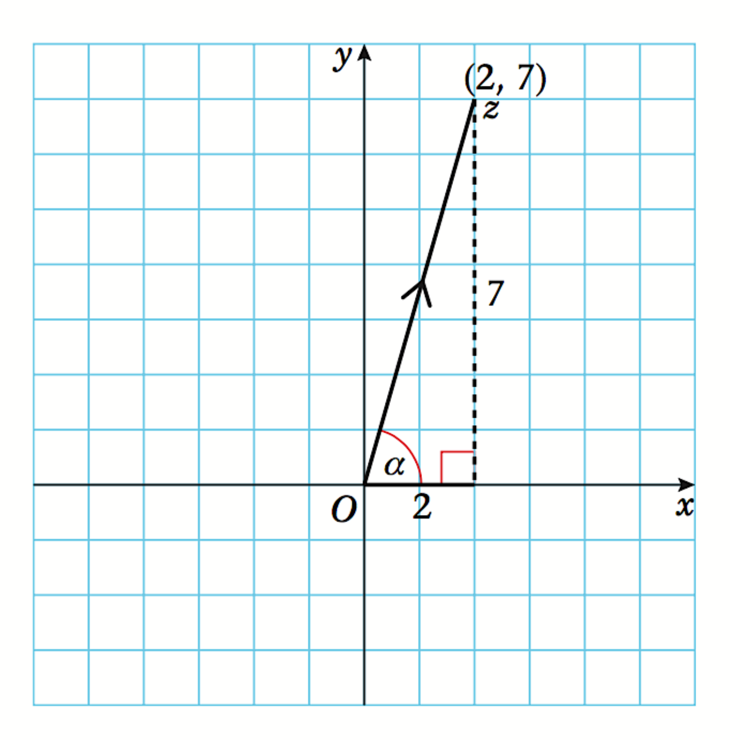
- The argument θ of a complex number is usually given in radians.

It is important to remember that the position of the complex number on the Argand diagram (the quadrant in which it appears) will determine whether its argument is positive or negative and whether its argument is acute or obtuse.

The following examples illustrate this.

Example 22

Find, to two decimal places, the modulus and argument (in radians) of $z = 2 + 7\text{i}$.



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |2 + 7\text{i}| = \sqrt{2^2 + 7^2} = \sqrt{53} = 7.28$ (2 d.p.)

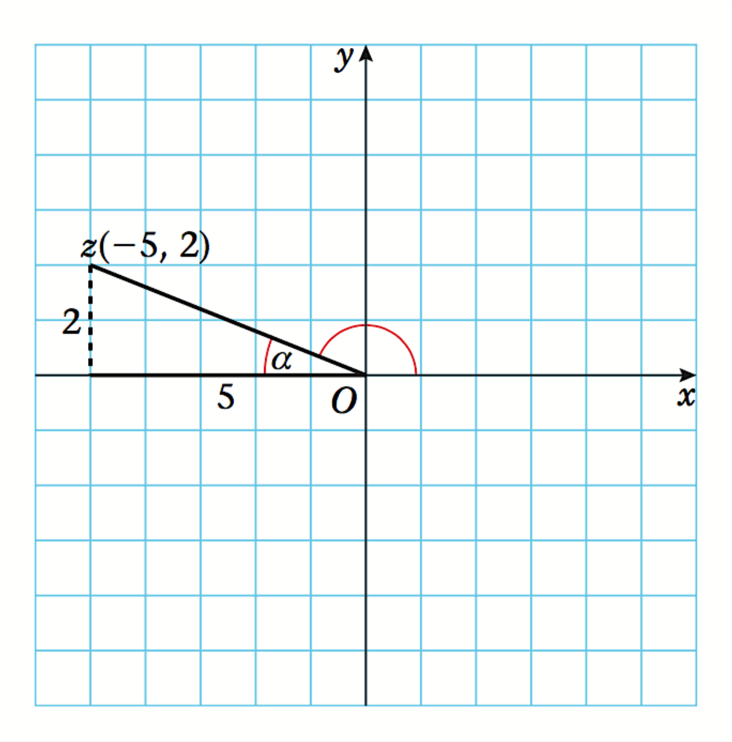
Argument: $\tan\alpha = \dfrac{7}{2}$ $\quad\alpha = 1.2924\ldots$ radians

$\arg z = 1.29$ radians (2 d.p.)

Here z is in the first quadrant, so this angle is the required argument (measured anticlockwise from the positive real axis).

Example 23

Find, to two decimal places, the modulus and argument (in radians) of $z = -5 + 2\text{i}$.



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |-5 + 2\text{i}| = \sqrt{(-5^2) + 2^2} = \sqrt{29} = 5.39$ (2 d.p.)

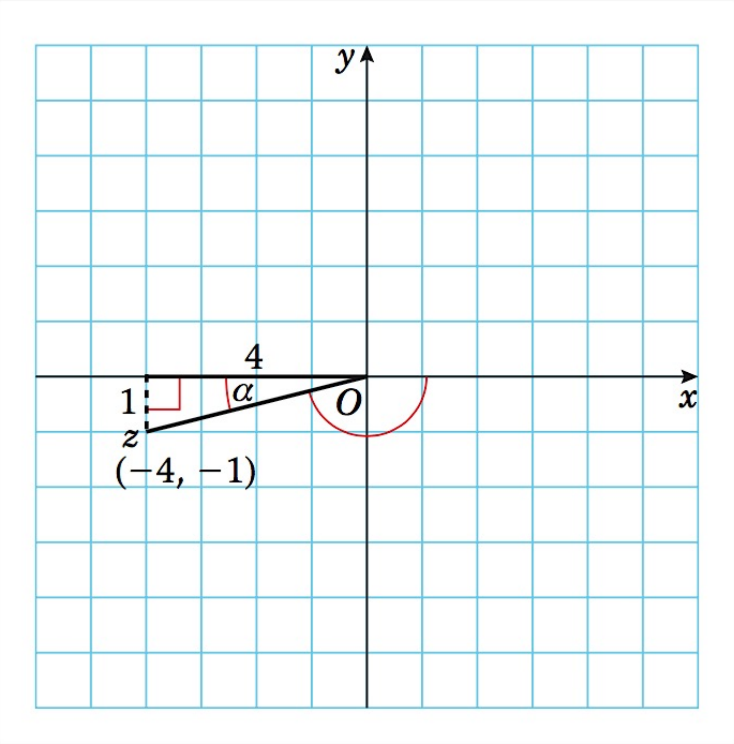
Argument: $\tan \alpha = \frac{2}{5}$ $\quad \alpha = 0.3805\ldots$ radians

$\arg z = (\pi - 0.3805) = 2.76$ radians (2 d.p.)

Here z is in the second quadrant, so the required argument is $(\pi - \alpha)$ (measured anticlockwise from the positive real axis).

Example 24

Find, to two decimal places, the modulus and argument (in radians) of $z = -4 - \mathrm{i}$.



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |-4 - \mathrm{i}| = \sqrt{(-4)^2 + (-1^2)} = \sqrt{17} = 4.12$ (2 d.p.)

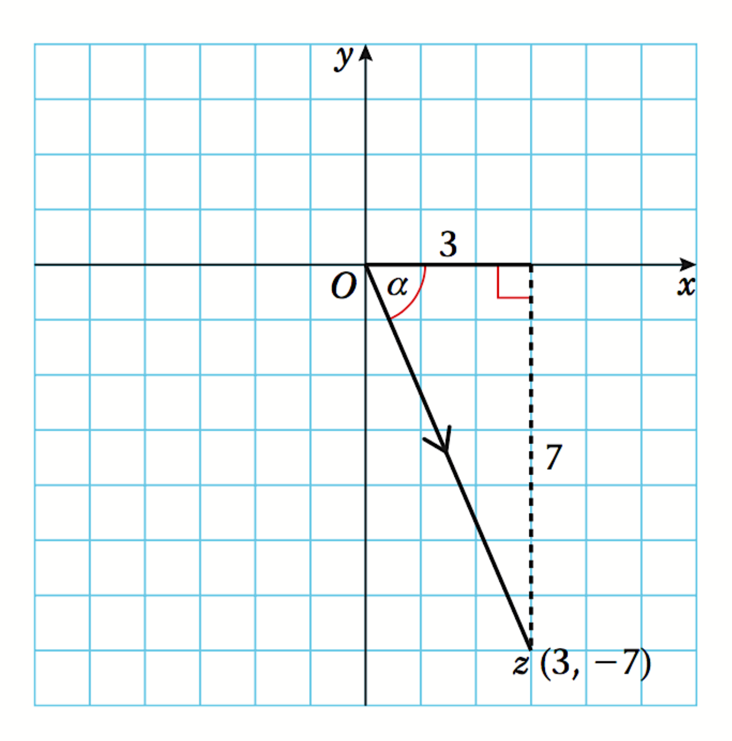
Argument: $\tan \alpha = \frac{1}{4}$ $\alpha = 0.2449\ldots$ radians

$\arg z = -(\pi - 0.2449) = -2.90$ radians (2 d.p.)

Here z is in the third quadrant, so the required argument is $-(\pi - \alpha)$ (clockwise from the positive real axis is negative).

Example 25

Find, to two decimal places, the modulus and argument (in radians) of $z = 3 - 7i$.



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |3 - 7i| = \sqrt{3^2 + (-7)^2} = \sqrt{58} = 7.62$ (2 d.p.)

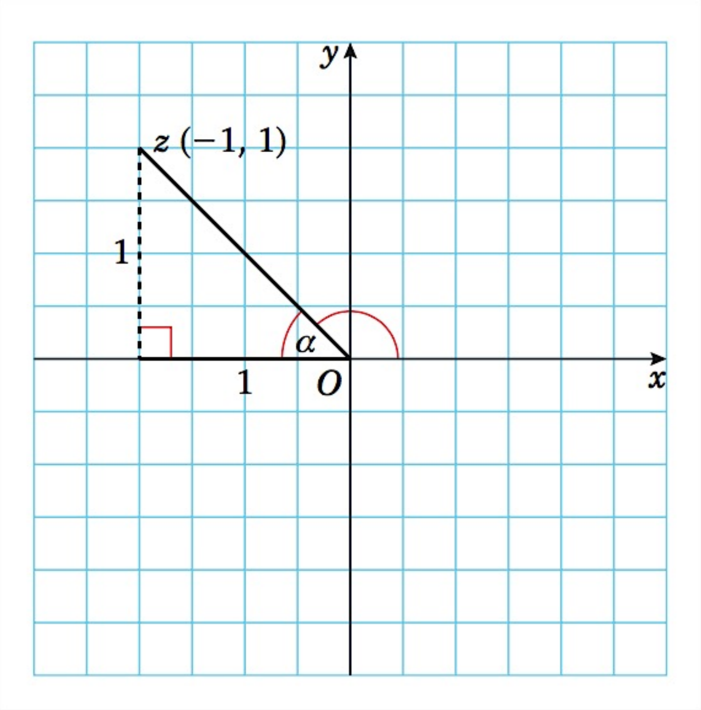
Argument: $\tan \alpha = \frac{7}{3}$ $\quad \alpha = 1.1659\ldots$ radians

$\arg z = -1.17$ radians (2 d.p.)

Here z is in the fourth quadrant, so the required argument is $-\alpha$ (clockwise from the positive real axis is negative).

Example 26

Find the exact values of the modulus and argument (in radians) of $z = -1 + \mathrm{i}$.



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |-1 + \mathrm{i}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

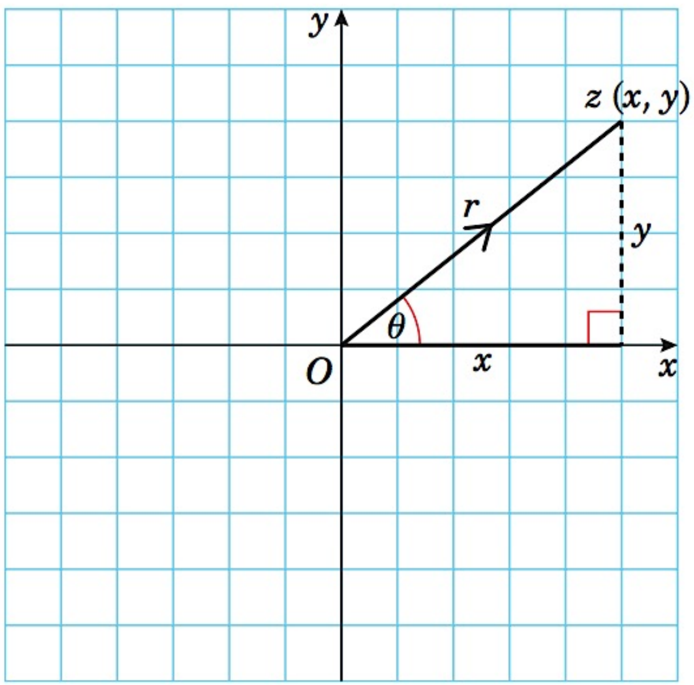
Argument: $\tan \alpha = \frac{1}{1} \qquad \alpha = \frac{\pi}{4}$

$\arg z = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}$

Here z is in the second quadrant, so the required argument is $(\pi - \alpha)$ (measured anticlockwise from the positive real axis).

You can find the modulus–argument form of the complex number z.

- The **modulus–argument** form of the complex number $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$ where r is a positive real number and θ is an angle such that $-\pi < \theta \leqslant \pi$ (or $-180° < \theta \leqslant 180°$)



From the right-angled triangle, $x = r\cos\theta$ and $y = r\sin\theta$.

This is correct for a complex number in any of the Argand diagram quadrants.

For complex numbers z_1 and z_2, $|z_1z_2| = |z_1||z_2|$.

Here is a proof of the above result.

Let $|z_1| = r_1$, arg $z_1 = \theta_1$ and $|z_2| = r_2$, arg $z_2 = \theta_2$, so

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2).$$

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2) = r_1r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

But $(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) = \cos(\theta_1 + \theta_2)$ and $(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) = \sin(\theta_1 + \theta_2)$

So $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)]$

You can see that this gives z_1z_2 in modulus-argument form, with $|z_1z_2| = r_1\,r_2$.

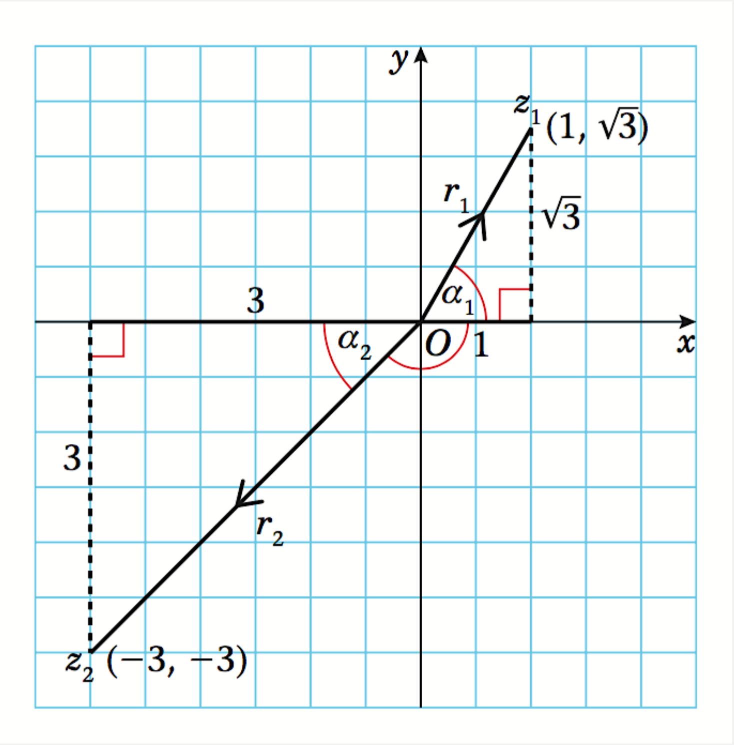
So $|z_1z_2| = r_1\,r_2 = |z_1||z_2|$

(Also, in fact, $\arg(z_1z_2) = \theta_1 + \theta_2$)

Example 27

a Express the numbers $z_1 = 1 + i\sqrt{3}$ and $z_2 = -3 - 3i$ in the form $r(\cos\theta + i\sin\theta)$.

b Write down the value of $|z_1 z_2|$.



Sketch the Argand diagram, showing the position of the numbers.

Modulus: $r_1 = |z_1| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

Argument: $\tan \alpha_1 = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \alpha_1 = \frac{\pi}{3}$

$\theta_1 = \arg z_1 = \frac{\pi}{3}$

z_1 is in the first quadrant, so this angle is the required argument (measured anticlockwise from the positive real axis).

Modulus: $r_2 = |z_2| = |-3 - 3i|$

$= \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{18} = \sqrt{9}\sqrt{2}$

$= 3\sqrt{2}$

Argument: $\tan \alpha_2 = \frac{3}{3} = 1 \qquad \alpha_2 = \frac{\pi}{4}$

$\theta_2 = \arg z_2 = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

z_2 is in the third quadrant, so the required argument is $-(\pi - \alpha_2)$ (clockwise from the positive real axis is negative).

So $\quad z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

and $\quad z_2 = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

Using $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$, $|z_1 z_2| = r_1 r_2 = 2 \times 3\sqrt{2} = 6\sqrt{2}$

You can solve problems involving complex numbers.

- You can solve problems by equating real parts and imaginary parts from each side of an equation involving complex numbers.
- This technique can be used to find the square roots of a complex number.
- If $x_1 + \mathrm{i}y_1 = x_2 + \mathrm{i}y_2$, then $x_1 = x_2$ and $y_1 = y_2$.

Example 28

38 Given that $3 + 5i = (a + ib)(1 + i)$, where a and b are real, find the value of a and the value of b.

$$(a + ib)(1 + i) = a(1 + i) + ib(1 + i)$$
$$= a + ai + bi - b$$
$$= (a - b) + i(a + b)$$

So $(a - b) + i(a + b) = 3 + 5i$

i $a - b = 3$

Equate the real parts from each side of the equation.

ii $a + b = 5$

Equate the imaginary parts from each side of the equation.

Adding **i** and **ii**: $2a = 8$

$a = 4$

Substituting into equation **i**:

Solve equations **i** and **ii** simultaneously.

$4 - b = 3$

$b = 1$

Example 29

Find the square roots of $3 + 4\mathrm{i}$.

Suppose the square root of $3 + 4\mathrm{i}$ is $a + \mathrm{i}b$,

where a and b are real.

Then $(a + \mathrm{i}b)^2 = 3 + 4\mathrm{i}$

$$(a + \mathrm{i}b)(a + \mathrm{i}b) = 3 + 4\mathrm{i}$$

$$a(a + \mathrm{i}b) + \mathrm{i}b(a + \mathrm{i}b) = 3 + 4\mathrm{i}$$

$$a^2 + ab\mathrm{i} + ab\mathrm{i} - b^2 = 3 + 4\mathrm{i}$$

$$(a^2 - b^2) + 2ab\mathrm{i} = 3 + 4\mathrm{i}$$

i $a^2 - b^2 = 3$ — Equate the real parts from each side of the equation.

ii $2ab = 4$ — Equate the imaginary parts from each side of the equation.

From **ii**: $b = \dfrac{4}{2a} = \dfrac{2}{a}$

Substituting into **i**: $a^2 - \dfrac{4}{a^2} = 3$ — Multiply throughout by a^2.

$$a^4 - 4 = 3a^2$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$ — This is a quadratic equation in a^2.

$$a^2 = 4 \text{ or } a^2 = -1$$

Since a is real, $a^2 = -1$ has no solutions.

Solutions are $a = 2$ or $a = -2$.

Substituting back into $b = \frac{2}{a}$:

When $a = 2$, $b = 1$

When $a = -2$, $b = -1$

So the square roots are $2 + i$ and $-2 - i$

The square roots of $3 + 4i$ are $\pm(2 + i)$.

You can solve some types of polynomial equations with real coefficients.

- You know that, if the roots α and β of a quadratic equation are complex, α and β are always a **complex conjugate pair.**
- Given one **complex root** of a quadratic equation, you can find the equation.
- Complex roots of a polynomial equation with real coefficients occur in **conjugate pairs.**

Example 30

$7 + 2i$ is one of the roots of a quadratic equation. Find the equation.

The other root is $7 - 2i$

The roots are a conjugate pair.

The equation with roots α and β is $(x - \alpha)(x - \beta) = 0$

See Example 14 for another method.

$(x - (7 + 2i))(x - (7 - 2i)) = 0$

$x^2 - x(7 - 2i) - x(7 + 2i) + (7 + 2i)(7 - 2i) = 0$

$x^2 - 7x + 2ix - 7x - 2ix + 49 - 14i + 14i - 4i^2 = 0$

$x^2 - 14x + 49 + 4 = 0$

$x^2 - 14x + 53 = 0$

- An equation of the form $ax^3 + bx^2 + cx + d = 0$ is called a **cubic equation**, and has three roots.

Example 31

Show that $x = 2$ is a solution of the cubic equation $x^3 - 6x^2 + 21x - 26 = 0$.
Hence solve the equation completely.

For $x = 2$, $x^3 - 6x^2 + 21x - 26 = 8 - 24 + 42 - 26 = 0$

So $x = 2$ is a solution of the equation, so $x - 2$ is a factor of $x^3 - 6x^2 + 21x - 26$

$$
\begin{array}{r l}
 & x^2 - 4x + 13 \\
x - 2 & \overline{)\, x^3 - 6x^2 + 21x - 26} \\
 & \underline{x^3 - 2x^2} \\
 & \quad -4x^2 + 21x \\
 & \quad \underline{-4x^2 + 8x} \\
 & \qquad\qquad 13x - 26 \\
 & \qquad\qquad \underline{13x - 26} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

Use long division (or another method) to find the quadratic factor.

$x^3 - 6x^2 + 21x - 26 = (x - 2)(x^2 - 4x + 13) = 0$

Solving $x^2 - 4x + 13 = 0$

The other two roots are found by solving the quadratic equation.

$x^2 - 4x = (x - 2)^2 - 4$

$x^2 - 4x + 13 = (x - 2)^2 - 4 + 13 = (x - 2)^2 + 9$

Solve by completing the square. Alternatively, you could use the quadratic formula.

$(x - 2)^2 + 9 = 0$

$(x - 2)^2 = -9$

$x - 2 = \pm\sqrt{(-9)} = \pm 3i$

$x = 2 \pm 3i$

The quadratic equation has complex roots, which must be a conjugate pair.

$x = 2 + 3i, x = 2 - 3i$

So the 3 roots of the equation are 2, $2 + 3i$, and $2 - 3i$.

Note that, for a cubic equation,

either **i** all three roots are real,

or **ii** one root is real and the other two roots form a complex conjugate pair.

Example 32

Given that -1 is a root of the equation $x^3 - x^2 + 3x + k = 0$,

a find the value of k,

b find the other two roots of the equation.

a If -1 is a root,

$(-1)^3 - (-1)^2 + 3(-1) + k = 0$

$-1 - 1 - 3 + k = 0$

$k = 5$

b -1 is a root of the equation, so $x + 1$ is a factor of $x^3 - x^2 + 3x + 5$.

$$
\begin{array}{r}
x^2 - 2x + 5 \\
x + 1 \overline{\smash{)}\, x^3 - x^2 + 3x + 5} \\
\underline{x^3 + x^2} \qquad\qquad\quad \\
-2x^2 + 3x \qquad \\
\underline{-2x^2 - 2x} \qquad \\
5x + 5 \\
\underline{5x + 5} \\
0
\end{array}
$$

Use long division (or another method) to find the quadratic factor.

$x^3 - x^2 + 3x + 5 = (x + 1)(x^2 - 2x + 5) = 0$

Solving $x^2 - 2x + 5 = 0$

The other two roots are found by solving the quadratic equation.

$x^2 - 2x = (x - 1)^2 - 1$

$x^2 - 2x + 5 = (x - 1)^2 - 1 + 5 = (x - 1)^2 + 4$

Solve by completing the square. Alternatively, you could use the quadratic formula.

$(x - 1)^2 + 4 = 0$

$(x - 1)^2 = -4$

$x - 1 = \pm\sqrt{(-4)} = \pm 2i$

$x = 1 \pm 2i$

The quadratic equation has complex roots, which must be a conjugate pair.

$x = 1 + 2i,\ x = 1 - 2i$

So the other two roots of the equation are $1 + 2i$ and $1 - 2i$.

- An equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ is called a **quartic equation**, and has four roots.

Example 33

Given that $3 + i$ is a root of the quartic equation $2x^4 - 3x^3 - 39x^2 + 120x - 50 = 0$, solve the equation completely.

Another root is $3 - i$.

Complex roots occur in conjugate pairs.

The equation with roots α and β is $(x - \alpha)(x - \beta) = 0$

$(x - (3 + i))(x - (3 - i)) = 0$

$x^2 - x(3 - i) - x(3 + i) + (3 + i)(3 - i) = 0$

$x^2 - 3x + ix - 3x - ix + 9 - 3i + 3i - i^2 = 0$

$x^2 - 6x + 9 + 1 = 0$

$x^2 - 6x + 10 = 0$

So $x^2 - 6x + 10$ is a factor of $2x^4 - 3x^3 - 39x^2 + 120x - 50$.

$$(x^2 - 6x + 10)(ax^2 + bx + c) = 2x^4 - 3x^3 - 39x^2 + 120x - 50$$

Consider $2x^4$

The only x^4 term in the expansion is $x^2 \times ax^2$, so $a = 2$.

$$(x^2 - 6x + 10)(2x^2 + bx + c) = 2x^4 - 3x^3 - 39x^2 + 120x - 50$$

It is possible to factorise 'by inspection' by considering each term of the quartic separately.

Consider $-3x^3$

The x^3 terms in the expansion are $x^2 \times bx$ and $-6x \times 2x^2$,

so $bx^3 - 12x^3 = -3x^3$

$b - 12 = -3$

so $b = 9$.

$(x^2 - 6x + 10)(2x^2 + 9x + c) = 2x^4 - 3x^3 - 39x^2 + 120x - 50$

Consider -50

The only constant term in the expansion is $10 \times c$, so $c = -5$.

$$2x^4 - 3x^3 - 39x^2 + 120x - 50 = (x^2 - 6x + 10)(2x^2 + 9x - 5)$$

You can check this by considering the x and x^2 terms in the expansion.

Solving $2x^2 + 9x - 5 = 0$

$(2x - 1)(x + 5) = 0$

$x = \frac{1}{2}, x = -5$

So the roots of $2x^4 - 3x^3 - 39x^2 + 120x - 50 = 0$ are

$\frac{1}{2}, -5, 3 + \text{i}$ and $3 - \text{i}$

Note that, for a quartic equation,

either **i** all four roots are real,

or **ii** two roots are real and the other two roots form a complex conjugate pair,

or **iii** two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

Example 34

Show that $x^2 + 4$ is a factor of $x^4 - 2x^3 + 21x^2 - 8x + 68$.
Hence solve the equation $x^4 - 2x^3 + 21x^2 - 8x + 68 = 0$.

Using long division:

$$\begin{array}{r|lllll}
 & x^2 & - 2x & + 17 & & \\ \cline{2-6}
x^2 + 4 & x^4 & - 2x^3 & + 21x & - 8x & + 68 \\
 & x^4 & & + 4x^2 & & \\ \cline{2-4}
 & & - 2x^3 & + 17x^2 & - 8x & \\
 & & - 2x^3 & & - 8x & \\ \cline{3-5}
 & & & 17x^2 & & + 68 \\
 & & & 17x^2 & & + 68 \\ \cline{4-6}
 & & & & & 0
\end{array}$$

So $x^4 - 2x^3 + 21x^2 - 8x + 68 = (x^2 + 4)(x^2 - 2x + 17) = 0$

It is also possible to factorise 'by inspection' by considering each term of the quartic separately, as in Example 33.

Either $x^2 + 4 = 0$ or $x^2 - 2x + 17 = 0$

Solving $x^2 + 4 = 0$

$x^2 = -4$

$x = \pm\sqrt{(-4)} = \pm\sqrt{(4 \times -1)} = \pm\sqrt{4}\sqrt{(-1)} = \pm 2i$

Solving $x^2 - 2x + 17 = 0$

$x^2 - 2x = (x - 1)^2 - 1$

$x^2 - 2x + 17 = (x - 1)^2 - 1 + 17 = (x - 1)^2 + 16$

$(x - 1)^2 + 16 = 0$

$(x - 1)^2 = -16$

$x - 1 = \pm\sqrt{(-16)} = \pm 4i$

$x = 1 \pm 4i$

So the roots of $x^4 - 2x^3 + 21x^2 - 8x + 68 = 0$ are

$2i, -2i, 1 + 4i$ and $1 - 4i$

Solve by completing the square. Alternatively, you could use the quadratic formula.

Summary of key points

1 $\sqrt{(-1)} = \text{i}$ and $\text{i}^2 = -1$.

2 An imaginary number is a number of the form $b\text{i}$, where b is a real number ($b \in \mathbb{R}$).

3 A complex number is a number of the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

4 For the complex number $a + b\text{i}$, a is called the real part and b is called the imaginary part.

5 The complex number $z^* = a - b\text{i}$ is called the complex conjugate of the complex number $z = a + b\text{i}$.

6 If the roots α and β of a quadratic equation are complex, α and β will always be a complex conjugate pair.

7 The complex number $z = x + \text{i}y$ is represented on an Argand diagram by the point (x, y), where x and y are Cartesian coordinates.

8 The complex number $z = x + \text{i}y$ can also be represented by the vector $\overrightarrow{OP}$, where O is the origin and P is the point (x, y) on the Argand diagram.

9 Addition of complex numbers can be represented on the Argand diagram by the addition of their respective vectors on the diagram.

10 The modulus of the complex number $z = x + \text{i}y$ is given by $\sqrt{x^2 + y^2}$.

11 The modulus of the complex number $z = x + \text{i}y$ is written as r or $|z|$ or $|x + \text{i}y|$, so

$$r = \sqrt{x^2 + y^2}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|x + \text{i}y| = \sqrt{x^2 + y^2}$$

12 The modulus of any non-zero complex number is positive.

13 The argument arg z of the complex number $z = x + iy$ is the angle θ between the positive real axis and the vector representing z on the Argand diagram.

14 For the argument θ of the complex number $z = x + iy$, $\tan\theta = \frac{y}{x}$.

15 The argument θ of any complex number is such that $-\pi < \theta \leqslant \pi$ (or $-180° < \theta \leqslant 180°$). (This is sometimes referred to as the principal argument.)

16 The modulus–argument form of the complex number $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$. [$r$ is a positive real number and θ is an angle such that $-\pi < \theta \leqslant \pi$ (or $-180° < \theta \leqslant 180°$)]

17 For complex numbers z_1 and z_2, $|z_1 z_2| = |z_1||z_2|$.

18 If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.

19 An equation of the form $ax^3 + bx^2 + cx + d = 0$ is called a cubic equation, and has three roots.

20 For a cubic equation, either

a all three roots are real, or

b one root is real and the other two roots form a complex conjugate pair.

21 An equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ is called a quartic equation, and has four roots.

22 For a quartic equation, either

a all four roots are real, or

b two roots are real and the other two roots form a complex conjugate pair, or

c two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

Part II

You can express a complex number in the form $z = r(\cos\theta + i\sin\theta)$

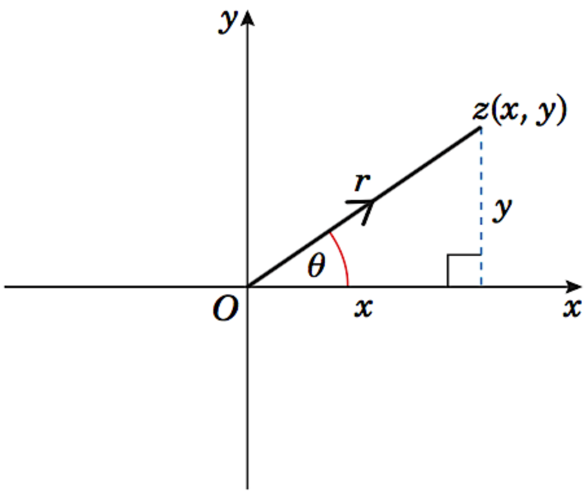
The **modulus–argument form** of the complex number $z = x + iy$ is

$$z = r(\cos\theta + i\sin\theta)$$

It is important for you to remember this formula.

where

- r, a positive real number, is called the modulus and
- θ, an angle such that when $-\pi < \theta \leqslant \pi$, θ is called the principal argument.



From the right-angled triangle,

$x = r\cos\theta$ and and $y = r\sin\theta$.

$r = |z| = \sqrt{x^2 + y^2}$

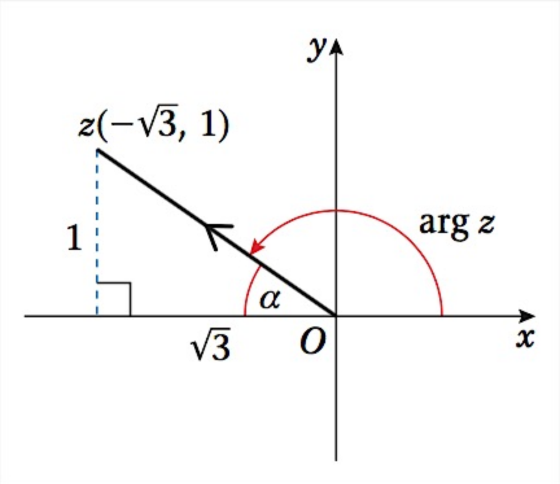
Note that θ, the argument, is not unique. The argument of z could also be $\theta \pm 2\pi$, $\theta \pm 4\pi$, etc.

To avoid duplication of θ, we usually quote θ in the range $-\pi < \theta \leqslant \pi$ and refer to it as the principal argument, 'arg', ie. $\theta = \arg z$.

$z = r(\cos\theta + i\sin\theta)$ is correct for a complex number in any of the Argand diagram quadrants.

Example 1

Express $z = -\sqrt{3} + \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the second quadrant so the required argument is $(\pi - \alpha)$.

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

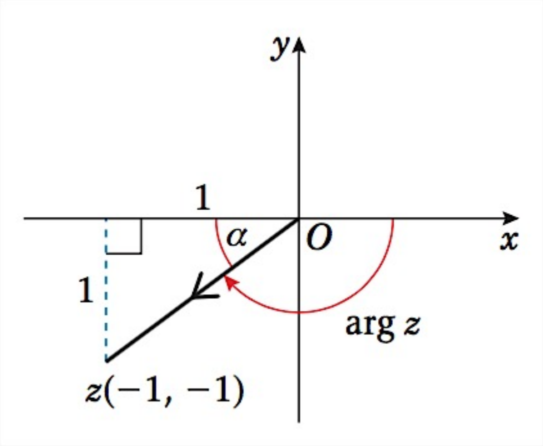
Find r and θ.

Therefore, $z = 2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$

Apply $z = r(\cos\theta + \mathrm{i}\sin\theta)$

Example 2

Express $z = -1 - \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the third quadrant so the required argument is $-\pi + \alpha$.

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{1}{1}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

Find r and θ.

Therefore, $z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$

Apply $z = r(\cos\theta + \mathrm{i}\sin\theta)$

You can express a complex number in the form $z = re^{i\theta}$.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\dots + \frac{(-1)^r\,\theta^{2r}}{(2r)!} + \dots\dots \qquad ①$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\dots + \frac{(-1)^r\,\theta^{2r+1}}{(2r+1)!} + \dots\dots \qquad ②$$

Also, for $x \in$!, the series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\dots + \frac{x^r}{r!} + \dots\dots$$

It can be proved that the series expansion for e^x is also true if x is replaced by a complex number. If you replace x in e^x by $i\theta$ the series expansion becomes

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots\dots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \dots\dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \dots\dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

By comparing this series expansion with those of ① and ② you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as Euler's relation.
It is important for you to remember this result.

You can now use Euler's relation to rewrite $z = r(\cos\theta + i\sin\theta)$ as

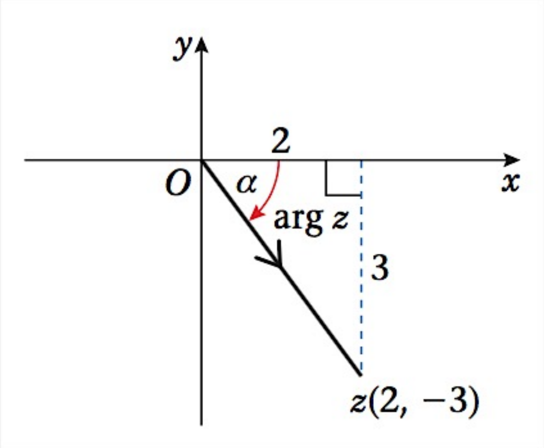
$$z = re^{i\theta}$$

This is the exponential form of the complex number z.

where $r = |z|$ and $\theta = \arg z$.

Example 3

Express $z = 2 - 3\text{i}$ in the form $re^{\text{i}\theta}$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the fourth quadrant so the required argument is $-\alpha$.

$r = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

$\theta = \arg z = -\tan^{-1}\left(\frac{3}{2}\right) = -0.98$ (2 d.p.)

Find r and θ.

Therefore, $z = \sqrt{13}\, e^{-0.98\text{i}}$

Apply $z = re^{\text{i}\theta}$.

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta$$

Example 4

Express

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + \text{i}\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - \text{i}\sin\frac{\pi}{8}\right)$ in the form $re^{\text{i}\theta}$, where $-\pi < \theta \leqslant \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + \text{i}\sin\frac{\pi}{10}\right)$

Compare with $r(\cos\theta + \text{i}\sin\theta)$.

So, $r = \sqrt{2}$ and $\theta = \frac{\pi}{10}$.

Therefore, $z = \sqrt{2}\,e^{\frac{\pi \text{i}}{10}}$

Apply $z = re^{\text{i}\theta}$.

b $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

$z = 5\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$

So, $r = 5$ and $\theta = -\frac{\pi}{8}$.

Therefore, $z = 5e^{-\frac{\pi i}{8}}$

Apply $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Compare with $r(\cos\theta + i\sin\theta)$.

Apply $z = re^{i\theta}$.

Example 5

Express $z = \sqrt{2}\,e^{\frac{3\pi i}{4}}$ in the form $x + iy$, where $x \in \square$ and $y \in \square$.

$$z = \sqrt{2}\,e^{\frac{3\pi i}{4}}$$

Compare with $re^{i\theta}$.

So, $r = \sqrt{2}$ and $\theta = \dfrac{3\pi}{4}$.

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Apply $r(\cos\theta + i\sin\theta)$.

$$= \sqrt{2}\ -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Apply $\cos\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}}$ and $\sin\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}}$.

Therefore, $z = -1 + i$

Simplify.

Example 6

Express $z = 2e^{\frac{23\pi i}{5}}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

$$z = 2e^{\frac{23\pi i}{5}}$$

Compare with $re^{i\theta}$.

So, $r = 2$ and $\theta = \dfrac{23\pi}{5}$.

$$\theta = \frac{23\pi}{5} \rightarrow \frac{13\pi}{5} \rightarrow \frac{3\pi}{5}$$

Continue to subtract 2π from θ until $2\pi < \theta \leqslant \pi$.

$$z = \sqrt{2}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

Apply $z = r(\cos\theta + i\sin\theta)$.

Example 7

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$$e^{i\theta} = \cos\theta + i\sin\theta \quad ①$$

$$e^{i(-\theta)} = e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

So, $e^{-i\theta} = \cos\theta - i\sin\theta \quad ②$

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Adding ① and ② gives

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

Divide both sides by 2.

Hence, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, as required.

You need to know how multiplying and dividing affects both the modulus and argument of the resulting complex number.

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2 \quad ④$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2 \quad ⑤$$

$$\cos^2\theta_2 + \sin^2\theta_2 = 1 \quad ⑥$$

Multiplying complex numbers z_1 and z_2

If $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\begin{aligned}
z_1z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 + \mathrm{i}^2\sin\theta_1\sin\theta_2)\\
&= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\
&= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\
&= r_1r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)), \text{ using identities ④ and ⑤.}
\end{aligned}$$

Therefore the complex number $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$ is in a **modulus–argument form** and has modulus r_1r_2 and argument $\theta_1 + \theta_2$.

Also, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$\begin{aligned} z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\ &= r_1 r_2 e^{i\theta_1 + i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Therefore the complex number $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ is in an **exponential form** and has modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$.

Dividing a complex number z_1 by a complex number z_2

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos\theta_2\cos\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin\theta_2\sin\theta_2)} \\ &= \frac{r_1((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)), \text{ using identities ④, ⑤ and ⑥.} \end{aligned}$$

Therefore the complex number $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$ is in **modulus–argument form** and has modulus $\frac{r_1}{r_2}$ and argument $\theta_1 - \theta_2$.

Also, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\
&= \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2} \\
&= \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2} \\
&= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}
\end{aligned}
$$

Therefore the complex number $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$ is in an **exponential form** and has modulus $\frac{r_1}{r_2}$ and argument $\theta_1 = \theta_2$.

In summary, you need to learn and apply the following results for complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

When you multiply z_1 by z_2
- you multiply their moduli and
- add their arguments as shown.

When you divide z_1 by z_2
- you divide their moduli and
- subtract their arguments as shown.

Example 8

Express $3\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ in the form $x + iy$.

$$3\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$= 3(4)\left(\cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + i\sin\left(\frac{5\pi}{12} + \frac{\pi}{12}\right)\right)$$

Apply the result,
$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

$$= 12\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Simplify.

$$= 12\,(0 + i(1))$$

Apply $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

$$= 12i$$

Example 9

Express $2\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right) \times 3\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$ in the form $x + iy$.

$$2\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right) \times 3\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$$

$$= 2\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right) \times 3\left(\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right)$$

$$= 2(3)\left(\cos\left(\frac{\pi}{15} - \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{15} - \frac{2\pi}{5}\right)\right)$$

$$= 6\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= 6\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= 3 - 3\sqrt{3}i$$

$z_2 = 3\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$ must be written in the form $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Apply the result, $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

Apply $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Example 10

Express $\dfrac{\sqrt{2}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)}{2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)}$ in the form $x + iy$.

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{12} - \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{12} - \frac{5\pi}{6}\right)\right)$$

By applying the result, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$.

$$= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

Simplify.

$$= \frac{\sqrt{2}}{2}\left(-\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right)$$

Apply $\cos\left(-\dfrac{3\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$ and $\sin\left(-\dfrac{3\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$.

$$= -\frac{1}{2} - \frac{1}{2}i$$

Summary of key points

1 A complex number, z, can be expressed in any one of three forms:

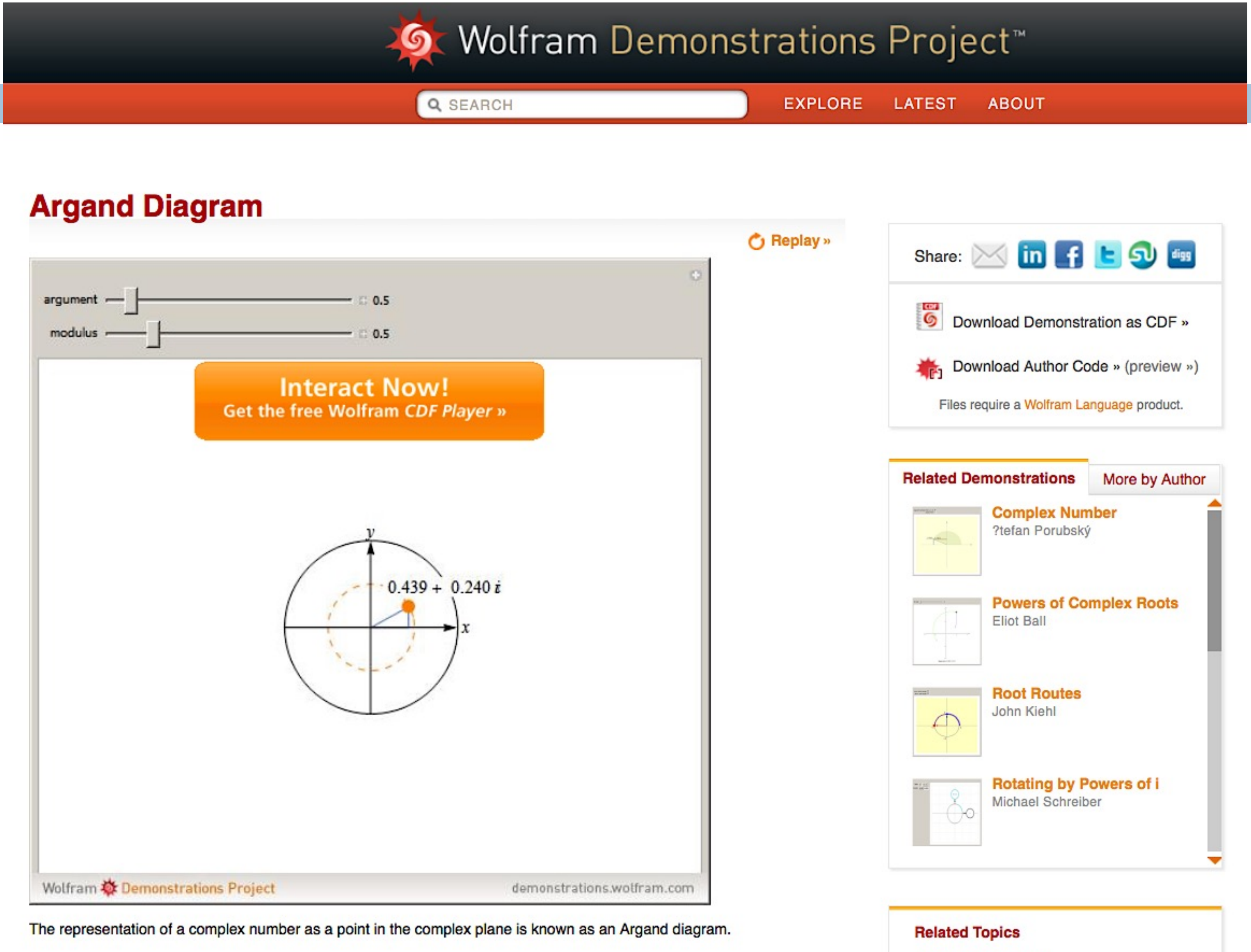
- $z = x = \mathrm{i}y$
- $z = r(\cos\theta + \mathrm{i}\sin\theta)$
- $z = r\mathrm{e}^{\mathrm{i}\theta}$

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z$.

2 For complex numbers $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$,

- $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$
- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
- $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2)$

Simulations



"Argand Diagram" from the Wolfram Demonstrations Project http://demonstrations.wolfram.com/ArgandDiagram

Applications to Physics

Alternating Current (AC)

Use of Complex Impedance

The handling of the impedance of an AC circuit with multiple components quickly becomes unmanageable if sines and cosines are used to represent the voltages and currents. A mathematical construct which eases the difficulty is the use of complex exponential functions. The basic parts of the strategy are as follows:

Math relationship underlying the technique	$e^{j\omega t} = \cos\omega t + j\sin\omega t$	Euler relation	Polar form of complex number
The real part of a complex exponential function can be used to represent an AC voltage or current.	$V = V_m \cos\omega t$, $I = I_m \cos(\omega t - \phi)$ represent by $\Rightarrow$ $V = V_m e^{j\omega t}$, $I = I_m e^{j[\omega t - \phi]}$		
The impedance can then be expressed as a complex exponential.	$Z = \frac{V_m}{I_m} e^{-j\phi} = R + jX$	Impedance combinations	Phasor diagrams
The impedance of the individual circuit elements can then be expressed as pure real or imaginary numbers.	$R \quad \frac{-j}{\omega C} \quad j\omega L$	RL and RC combinations	Example for parallel elements

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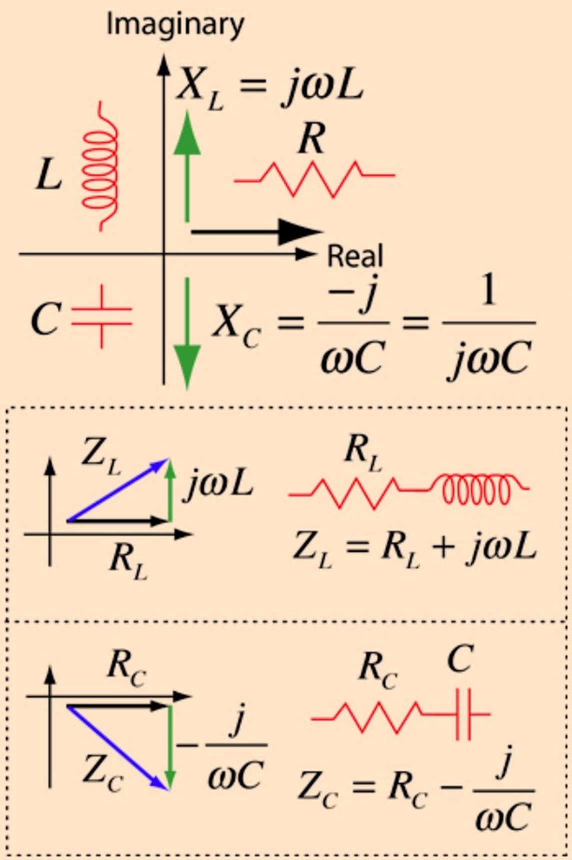
AC circuit concepts

http://hyperphysics.phy-astr.gsu.edu/hbase/electric/impcom.html#c1

Applications to Physics

Alternating Current (AC)

Complex Impedance for RL and RC



Using complex impedance is an important technique for handling multi-component AC circuits. If a complex plane is used with resistance along the real axis then the reactances of the capacitor and inductor are treated as imaginary numbers. For series combinations of components such as RL and RC combinations, the component values are added as if they were components of a vector. Shown here is the cartesian form of the complex impedance. They can also be written in polar form. Impedances in this form can be used as building blocks for calculating the impedances of combination circuits like the RLC parallel circuit.

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AC circuit concepts

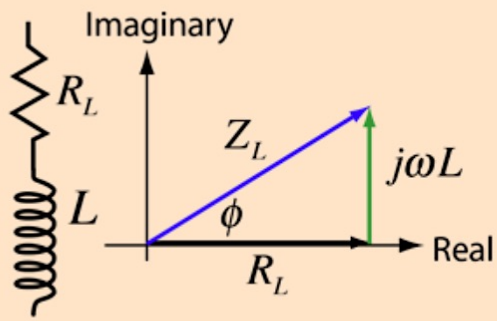
http://hyperphysics.phy-astr.gsu.edu/hbase/electric/impcom.html#c1

Applications to Physics

 Alternating Current (AC)

Complex Impedance for RL and RC

This depicts the phasor diagrams and complex impedance expressions for RL and RC circuits in polar form. They can also be expressed in cartesian form.

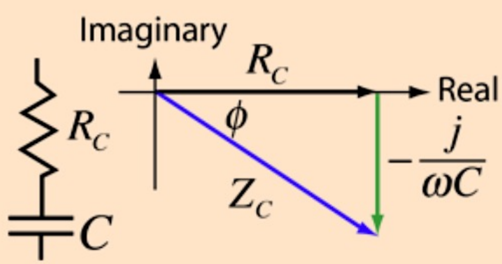


Cartesian form: $Z_L = R_L + j\omega L$

Polar form: $Z_L = |Z_L|e^{j\phi}$

where $|Z_L| = \sqrt{R_L^2 + \omega^2 L^2}$

$\phi = \tan^{-1}\dfrac{\omega L}{R_L}$



Cartesian form: $Z_C = R_C - \dfrac{j}{\omega C}$

Polar form: $Z_C = |Z_C|e^{j\phi}$

where $|Z_C| = \sqrt{R_C^2 + \left[\dfrac{-1}{\omega C}\right]^2}$

$\phi = \tan^{-1}\dfrac{-1}{\omega C R_C}$

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AC circuit concepts

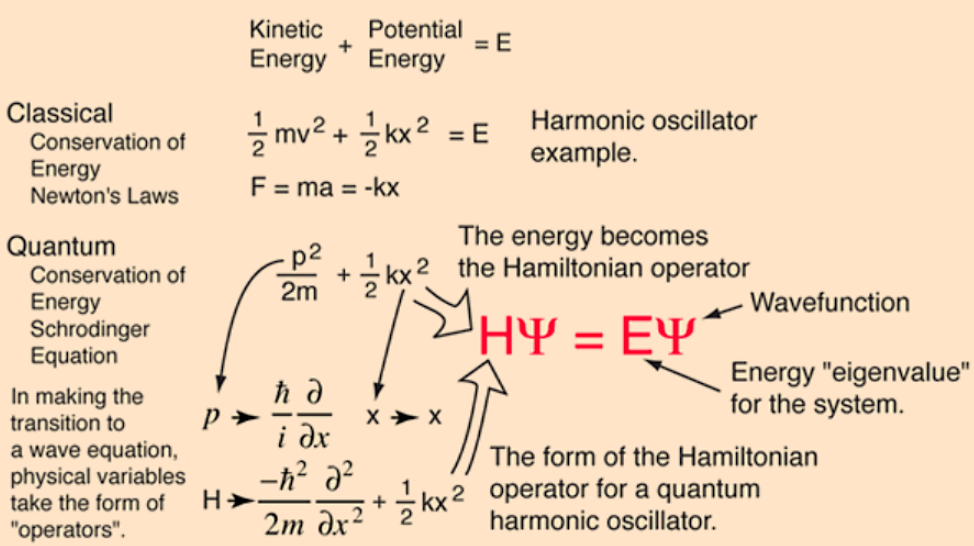
Impedance | Polar form of complex number

http://hyperphysics.phy-astr.gsu.edu/hbase/electric/impcom.html#c1

Applications to Physics

Schroedinger Equation

The Schroedinger equation plays the role of Newton's laws and conservation of energy in classical mechanics - i.e., it predicts the future behavior of a dynamic system. It is a wave equation in terms of the wavefunction which predicts analytically and precisely the probability of events or outcome. The detailed outcome is not strictly determined, but given a large number of events, the Schroedinger equation will predict the distribution of results.



The kinetic and potential energies are transformed into the Hamiltonian which acts upon the wavefunction to generate the evolution of the wavefunction in time and space. The Schroedinger equation gives the quantized energies of the system and gives the form of the wavefunction so that other properties may be calculated.

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Schroedinger equation concepts

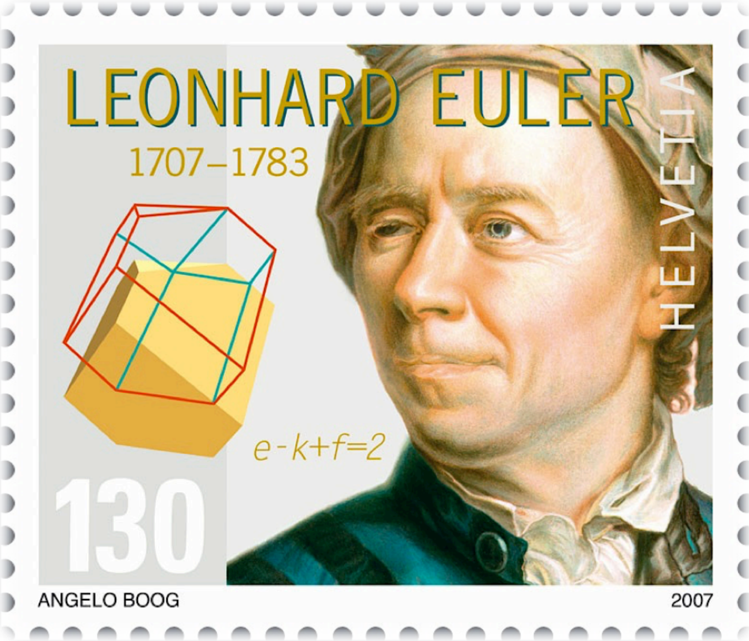
http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr.html#c1

Physicists/Mathematicians

Leonhard Euler (1707–1783)

Swiss mathematician. Euler's name is attached to every branch of mathematics. His prolific writing was not at all slowed down by his total blindness for the last seventeen years of his life. He could recall and mentally calculate long and complicated problems. Euler did not cease to calculate until he ceased to live – that day he was still talking about the calculation of the orbit of Uranus. Among the many new symbols Euler introduced were the signs:

$$i \text{ for } \sqrt{-1}$$
$$\sum \text{ for summation}$$
$$f(x) \text{ for function}$$
$$e \text{ for the base of natural logarithm}$$



http://lombokmusic.com/wp-content/uploads/2012/03/LEONHARD-EULER-.jpg

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